# CIRCLES ORTHOGONAL TO A GIVEN SPHERE

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In his dissertation Dr. Forbes* has discussed analytically the geometry of circles orthogonal to a given sphere and by means of coordinates has established the relation between this geometry and the ordinary geometry of the straight line. It is the object of this paper to develop the geometry of circles orthogonal to a given sphere by means of a transformation which transforms straight lines into circles orthogonal to a fixed sphere. By this method we are enabled to obtain the properties of the surface of singularities of the quadratic complex which we were unable to do by means of the analysis. The transformation used has been partially discussed by Laguerre † and Darboux ‡

**1. The Transformation.** In order to define this transformation we take a sphere of reference $S$ with center at $O$. Every plane $P$ determines with $S$ a pencil of spheres which contains two point spheres. Calling the centers of these point spheres $p$ and $p'$, *the transformation we wish to consider is the one which carries the plane $P$ over into the points $p$, $p'$*. We shall call $p$ and $p'$ the representative point pair of the plane $P$. It is well known that these two points are related to each other by inversion with respect to $S$.

* C. S. Forbes, Geometry of circles orthogonal to a given sphere, New York, 1904.

† *Œuvres*, vol. II, p. 54.

‡ *Sur une classe remarquable des courbes, etc.*, p. 123.

We begin by establishing the following property of this transformation.

*If the plane $P$ is revolved about a line $l$ the points $p$, $p'$ will generate a circle $C$ orthogonal to $S$ whose center lies on $l$ and whose plane passes through $O$ and is perpendicular to $l$.*

Since the centers of the spheres of the pencil formed by $S$ and $P$ lie on a line passing through $O$ and perpendicular to $P$ it follows that as $P$ is revolved about $l$ the centers of the point spheres $p$, $p'$ will lie in a plane passing through $O$ and perpendicular to $l$. This plane will cut $S$ in a great circle $C_1$ and will cut the book of planes having $l$ for axis in a pencil of lines $V$. It is evident that the points $p$, $p'$ are the centers of the point circles of the pencils determined by $C_1$ and the line of $V$ which corresponds to $P$. With the vertex of $V$ as center draw a circle $C$ orthogonal to $C_1$. Then since $C$ is orthogonal to $C_1$ and to each line of the pencil $V$ it is orthogonal to the pencil determined by $C_1$ and any line of $V$, and hence contains the points $p$ and $p'$ (by familiar properties of a pencil of circles). Hence the theorem is established. Conversely all the points of $C$ can be obtained as the centers of the point circles of one of the above pencils. For it is well known from the property of orthogonal circles that a line drawn through $O$ will cut $C$ in points $p$, $p'$ related by inversion with respect to $C_1$ and $p$ and $p'$ are the point circles of the pencil determined by $C_1$ and the line bisecting the chord joining $pp'$.

In view of the theorem just proved the line $l$ will be said to be transformed into the circle $C$.

From what has just been said it follows at once that:

*If the plane $P$ is pivoted about a point $O'$ the points $p$, $p'$ will generate a sphere $S'$ orthogonal to $S$ and having $O'$ for center.*

In this sense the point $O'$ will be said to be transformed into the sphere $S'$.

It is well known that the common circle $C$ of a pencil of spheres orthogonal to $S$ is itself orthogonal to $S$. From the theorem stated above the points of a line $l$ are transformed into a pencil of spheres orthogonal to $S$. The common circle $C$ of the pencil is orthogonal to $S$; its center lies on $l$ and its plane passes through $O$ (since this plane belongs to the pencil of spheres orthogonal to $S$); it is therefore the same circle which was obtained by transforming the planes containing $l$. The circle $C$ may therefore be considered either as the locus of the point pairs $p$, $p'$ obtained by transforming the planes containing $l$ or as the envelope of the spheres $S'$ obtained by transforming the points of $l$.

The results of the transformation may be tabulated as follows.

| *Original element* | *Transformed element* |
|---|---|
| A plane. | A pair of points related to each other by inversion with respect to $S$. |
| A line. | A circle orthogonal to $S$ having its center on the line and lying in a plane perpendicular to the line and passing through $O$. |
| A point. | A sphere orthogonal to $S$ and having the point for center. |
| Points on a line. | Spheres having a circle orthogonal to $S$ in common. |
| Planes having a line in common. | Point pairs lying on a circle orthogonal to $S$. |
| A plane and a point on it. | A point pair lying on a sphere orthogonal to $S$. |
| A plane and a line in it. | A point pair lying on a circle orthogonal to $S$. |
| A line and a point on it. | A circle orthogonal to $S$ lying on a sphere orthogonal to $S$. |

The exceptional elements of the transformation are minimum planes, the plane at infinity, points in the plane at infinity, and lines in the plane at infinity. The result of transforming these elements is given in the table below. These facts will be evident to those who are accustomed to reasoning about minimum planes, the plane at infinity, etc., and for those who are not so accustomed the proof will be given in the appendix.

| *Original element* | *Transformed element* |
|---|---|
| A minimum plane (a plane which cuts the circle at infinity in a single point). | The point where the plane touches the circle at infinity together with a definite point on the line joining $O$ to this point. |
| The plane at infinity. | The point $O$ together with any point whatever in the plane at infinity. |

| | |
|---|---|
| A point in the plane at infinity. | A plane passing through $O$ together with the plane at infinity itself. |
| A line in the plane at infinity. | A line passing through $O$ together with a line in the plane at infinity. |

From the properties of the transformation it follows at once that a line $l$ which is tangent to $S$ is transformed into the point circle whose center is at the point of tangency and whose plane is perpendicular to $l$ and passes through $O$.

Any configuration of lines is transformed into the corresponding configuration of circles orthogonal to $S$. Lines which intersect are transformed into circles which intersect in the representative point pair of their common plane. Thus we see that circles orthogonal to $S$ which intersect in one point intersect in a second point related to the first by inversion with respect to $S$. The $\infty^2$ lines in a plane are transformed into the $\infty^2$ circles orthogonal to $S$ which intersect in the representative point pair of the plane. The $\infty^2$ lines passing through a point $O'$ are transformed into the $\infty^2$ circles orthogonal to $S$ which lie on $S'$, the sphere into which $O'$ is transformed. It is easily seen that the centers of these circles lie on a second sphere which has $OO'$ for diameter.

If we apply the above transformation to the points of a non-developable surface $A$ we obtain a surface enveloped by spheres orthogonal to $S$ and having their centers on $A$. In order to find the points of contact of any enveloping sphere $R$ with the transformed surface it is necessary to find the limiting points of intersection of enveloping spheres as they approach coincidence with $R$. The points of $A$ which lie in a plane are transformed into spheres which have the representative point pair of the plane in common. Now as the plane approaches coincidence with the plane tangent to $A$ at the center of $R$ the points of intersection of the spheres approach coincidence with the representative point pair of the tangent plane. From which we see that we obtain the points of contact of any enveloping sphere $R$ by transforming the plane tangent to $A$ at the center of $R$. Therefore:

*The surfaces obtained by transforming the points of $A$ and the planes tangent to $A$ are identical.*

It is evident that the surfaces obtained by transforming the points of a curve and the tangent lines to the curve are identical.

We will now consider the cases when the surface $A$ is a scroll (ordinary ruled surface) or a developable. Since any plane containing a generator of a

scroll is tangent to the scroll the surfaces obtained by transforming the generators and by transforming the tangent planes are identical. The generators of the scroll are transformed into the circle generators of a circled surface. This surface cannot be generated as the envelope of a one parameter family of spheres, since it cannot be obtained by transforming a one parameter family of points. This is expressed by saying the surface is non-annular.

If the surface $A$ is a developable surface and is considered as generated by the tangent lines to a twisted curve it is transformed into a circled surface which can be generated as the envelope of a one parameter family of spheres; for the same surface can be obtained by transforming the points of the twisted curve. This surface is an annular surface. If the developable is considered as the envelope of the osculating planes to a twisted curve it is transformed into a second twisted curve (since it is obtained by transforming a one parameter family of planes), which is the locus of the limiting points of intersection of the circles of the annular surface above. For if we transform any three points of the original curve, the points of intersection of the circles common to the three spheres will be the representative point pair of the plane of the three points. Now as the three points approach coincidence the plane approaches the osculating plane of the curve and the circles common to the spheres approach circles of the annular surface.

It will be seen later that the order of the curve is $2n$ where $n$ is the order of the developable. This curve is analogous to the cuspidal edge of the developable. The tangent lines of a plane curve are transformed into the circle generators of an annular surface. All these circles have the representative point pair of the plane of the curve in common. These last surfaces play the same rôle in this geometry that cones do in the ordinary line geometry. In particular if the plane curve is a conic the circled surface, as will appear later, will be a binodal cyclide which plays an important part in what follows.

It is at once seen that a cone is transformed into a curve traced on the sphere into which the vertex of the cone is transformed. The elements of the cone are transformed into circles which envelope the curve, and the tangent planes of the cone are transformed into the points of the curve.

This is the transformation which Laguerre and Darboux used to obtain the cyclides from the quadric surfaces. Darboux* defines a point transformation by a combination of polar reciprocation with respect to $S$ and the trans-

* Loc. cit., p. 126.

formation described above. He derives the equations of this transformation and by means of them obtains the properties of the transformed surface. The properties of the surface obtained by applying the transformation under discussion can be obtained, however, without the use of equations.

**2. Transformation of surfaces.** A surface $A$ of class $n$ is transformed into a surface which has the circle at infinity as $n$-fold line. For, consider a pencil of $n$ minimum planes, by which we mean a pencil whose axis is a line in the plane at infinity which touches the circle at infinity; $n$ planes of this pencil will touch $A$, and (since the representative point pair of a minimum plane consists of the point where the plane touches the circle at infinity and a point on the line joining $O$ to this point) *each point on the circle at infinity will be an n-fold point on the transformed surface.*

We can now obtain the order of the transformed surface. The surface $A$ is of class $n$ and therefore $n$ planes of any pencil will touch $A$. Each of these tangent planes is tranformed into two points of the transformed surface. Moreover the axis of the pencil is transformed into a circle which contains the representative point pair of each plane of the pencil and in particular of each of the tangent planes. Hence this circle intersects the transformed surface in $2n$ finite points, and, since any circle cuts the circle at infinity in two points, we have still $2n$ points of intersection of the circle and the surface to reckon. The total number of points of intersection is therefore $4n$, and we have the theorem:

*The transformed surface is of order* $2n$ *and contains the circle at infinity as an n-fold line.*

We saw that the plane at infinity is transformed into $O$ together with any point in the plane at infinity (that is, into the plane at infinity itself). Therefore if a surface has the plane at infinity for tangent plane the transformed surface will contain the plane at infinity as a part, i. e., the surface will be factorable, the order of the residual surface will be reduced by one, and the number of times which the surface contains the circle at infinity will also be reduced by one. Hence *if the surface $A$ is tangent $r$ times to the plane at infinity the transformed surface will be of order* $2n - r$ *and will contain the circle at infinity as an* $(n - r)$*-fold line.*

If the axis of the pencil of minimum planes considered above touches $A$, two planes of the pencil will coincide and therefore it is easily seen that two sheets of the transformed surface will be tangent at this point. The number

of such points will be equal to the number of common tangents that can be drawn to the circle at infinity and the seetion of $A$ in the plane at infinity. Loria calls such points *cuspidal points.*

*Surfaces obtained by transforming scrolls.*

The straight lines on the ruled surface become circles on the transformed surface. The centers of these circles lie on the given ruled surface at the points of intersection of the generators and lines drawn from $O$ perpendicular to the generators. The order of the curve of centers will be equal to the number of generators of the ruled surface whose normals drawn from $O$ lie in a plane.

All lines whose normals drawn from $O$ lie in a plane $P$ can be arranged in plane pencils whose vertices lie in $P$ and such that the line drawn from $O$ to a vertex is normal to the plane of the corresponding pencil. All the lines of the pencil that pass through a fixed point not in $P$ generate a cone whose section in $P$ is a circle having $O$ and the foot of the perpendicular dropped from the point to $P$ as extremities of a diameter. The generators of a ruled surface establish a $(1, 1)$ correspondence between the points of any two general plane sections $c_1$ and $c_2$. If from each point of $c_1$ we draw the cone of lines whose normals from $O$ lie in $P$, $c_2$ will intersect it in $2n$ points and therefore a $(1, 2n)$ correspondence is established between the points of $c_1$ and $c_2$, and vice versa. Now let $A_0$, $B_0$ be the points where a generator cuts $c_1$ and $c_2$ respectively. To $A_0$ corresponds $2n$ points of $c_2$ as has just been shown; but $A_0$ corresponds to $B_0$ (by correspondence set up by generators of the surface); hence to $B_0$ correspond $2n$ points of $c_2$, from which we see that to each point of $c_2$ considered as a point on a generator correspond $2n$ points considered as points on the cones, and vice versa. Therefore a $(2n, 2n)$ correspondence is established between the points of $c_2$, and the coincidences in general give the number of generators of the ruled surface which are also elements of the cones. But at each of the $n$ intersections of $c_1$ and $c_2$ there are two coincidences which do not give elements of the cones, for such a point is the vertex of a cone and therefore two elements of the cone lie in the plane of $c_2$; these two elements correspond to the generator through the point but they do not coincide. In a $(2n, 2n)$ correspondence there are in all $4n$ coincidences but as $2n$ of these do not give elements of the cones we have:

*The locus of the centers of the circles of a circled surface obtained by*

*transforming a ruled surface of order $n$ is a curve of order $2n$.* It can easily be shown that the planes of the circles envelope a cone of order $n$. The locus of the centers of the circles into which the elements of a cone are transformed is a curve traced on a sphere. It has a multiple point at the vertex of the cone.

It is known from the theory of ruled surfaces that each generator of a ruled surface cuts $n-2$ other generators; that is, each generator cuts the nodal curve in $n-2$ points. Therefore in the transformed surface each circle generator cuts $n-2$ other generators, or it cuts the nodal curve in $2(n-2)$ points. Multiple generators on the ruled surface are transformed into multiple generators of the circled surface. A multiple rectilinear directrix of order $r$ is transformed into a circular directrix of order $n-r$, for any plane which contains the directrix will cut from the ruled surface $n-r$ generators, and therefore through each point pair of the circle into which the directrix is transformed will pass $n-r$ generators of the circled surface. Each point of the rectilinear directrix of the ruled surface is transformed into a sphere which contains the circle into which the directrix is transformed and also the circles into which the generators of the ruled surface which pass through the point are transformed. This is the complete intersection of the sphere with the circled surface. Hence, *a sphere which passes through a directrix of order $n-r$ on the circled surface cuts from it $r$ circular generators.*

To a multiple rectilinear directrix of the ruled surface corresponds a second nodal curve on the circled surface, viz.: the curve into which the developable containing two generators passing through the same point of the directrix is transformed. To a nodal curve on a ruled surface which is not a straight line corresponds a nodal curve on the circled surface obtained by transforming the developable formed by planes containing two generators through the same point of the nodal curve. A torsal generator of the ruled surface is transformed into a torsal generator of the circled surface.

**3. Congruences of circles.** To every congruence of straight lines corresponds by the transformation a congruence of circles orthogonal to $S$, and vice versa. We will define the order of the congruence as the number of circles of the congruence which pass through an arbitrary point pair, and the class of the congruence as the number of circles which lie on an arbitrary sphere orthogonal to $S$. Then it follows at once from the transformation that the order and class of a congruence of circles orthogonal to a given sphere are

equal respectively to the class and order of the congruence of straight lines formed by the axes of the circles.

*The focal surface.* The focal points of any curve of a congruence are defined as the points of the curve in which it is cut by the curves of the congruence which are infinitely close and which cut it. The focal surface of the congruence is defined as the locus of these focal points. In a congruence of circles orthogonal to $S$, intersecting circles will result from transforming lines of the line congruence which intersect, and since each line of the congruence is intersected by two lines of the congruence infinitely near, and these are transformed into intersecting circles, it follows that each circle of the congruence will be cut in two points by each of two circles infinitely near. Hence, *on each circle there are four focal points, two of which are the inverses of the other two with respect to* $S$. These points are obtained by transforming the planes which contain the given line and one of the lines infinitely close; but such planes are tangent to the focal surface of the line congruence. Hence *the focal surface of the circle congruence is obtained by transforming the focal surface of the line congruence.*

If the line congruence consists of all lines which cut two curves or which cut a given curve in two points, the focal surface degenerates into the curves. The focal surface of the corresponding circle congruence becomes the annular surfaces into which the curves are transformed. In particular, if the line congruence is linear the focal surface consists of two non-intersecting straight lines and the focal surface of the corresponding congruence of circles becomes the pencils of spheres into which the points of the lines are transformed, i. e., the circles enveloped by these spheres; therefore we may define a linear congruence of circles as all the circles which cut two fixed non-intersecting circles.

**4. The linear complex.** A complex of lines is transformed into a complex of circles and in particular a linear complex of lines is transformed into a linear complex of circles. By the transformation the configurations of the linear circle complex are connected with those of the linear line complex as follows:

| *Line complex* | *Circle complex* |
|---|---|
| Lines of the complex lying in the same plane pass through the same point called the pole of the plane. | Circles of the complex which pass through the same point pair lie on the same sphere called the polar sphere of the point pair. |

Lines which pass through the same point lie in the same plane called the polar plane of the point.

Circles which lie on the same sphere pass through the same point pair called the polar point pair of the sphere.

If a plane revolves about a line its pole describes a second line called the polar conjugate.

If a point pair describe a circle the polar spheres will envelope a second circle called the polar conjugate.

Every line of the complex which cuts a given line will also cut its conjugate polar.

Every circle of the complex which cuts a given circle will also cut its conjugate polar.

The line joining a point pair always passes through $O$, the center of $S$. We will call the anharmonic ratio of four point pairs, on the same circle, the anharmonic ratio of the four lines joining them to $O$. Then from the (1, 1) correspondence between the point pairs and their polar spheres we see that *the anharmonic ratio of four point pairs on the same circle is the same as the anharmonic ratio of their polar spheres.*

In the remainder of the paper considerable knowledge of the configurations of lines of both the linear and quadratic line complex is presupposed. By an application of the above transformation to these configurations we obtain the configurations of the linear and quadratic circle complex. The results of this transformation will be given, but space forbids giving more than a hint of the demonstration.

*The form of the linear complex.* The linear line complex remains invariant with respect to rotation about or translation along its axis. Through $O$ draw a plane $\pi$ normal to the axis of the line complex. By the transformation the axis will be transformed into a circle $C$ lying in $\pi$ and such that all the circles which cut it will lie in planes perpendicular to $\pi$ (since by this transformation the angle between two circles is the supplement of the angle between the corresponding lines). These planes form a pencil whose axis $l$ passes through $O$ and is parallel to the axis of the line complex. Planes which contain lines making the same angle with the axis of the line complex envelope a circular cylinder with the axis of the complex for axis. The lines in any one of these planes are transformed into circles lying in a plane which makes the same angle with $\pi$ that the lines made

with the axis of the complex; these planes then envelope a circular cone. The circles which lie in the same enveloping plane of this cone intersect in a point pair in $\pi$ and the locus of this point pair is a bicircular quartic. This quartic is the envelope of circles, in $\pi$, whose centers lie on the circle which $\pi$ cuts from the corresponding cylinder and which are orthogonal to $S$. Therefore *a linear circle complex consists of those circles which lie in planes enveloping a series of coaxial circular cones and whose points of intersection lie on a bicircular quartic.*

Analogous to Sylvester's method of constructing the linear line complex, the linear circle complex consists of all circles which intersect two corresponding circles of two projective pencils which have a common self-corresponding circle.

*Linear series.* Lines common to three linear line complexes form one generation of the hyperboloid; therefore circles common to three linear circle complexes will be the surface into which the hyperboloid is transformed. This is the generation of the cyclide given by Laguerre.*

Since four generators of each system of the hyperboloid are tangent to $S$, and to each of these corresponds two minimum lines, the 16 minimum lines of the cyclide are degenerate circles. The planes of these degenerate circles pass through $O$ and their centers (points of intersection of the minimum lines) lie on the sphero-quartic in which $S$ intersects the hyperboloid. Since each generator of one system of the cyclide intersects all the circles of the other system, each minimum line will intersect five others. The cyclide can be generated in five ways,† as above; therefore *the points of intersection of the minimum lines can be arranged in five groups of eight which lie on a sphero-quartic and such that the planes containing the pairs of lines of any group pass through a common point.* The locus of centers of the circles of one generation of the cyclide is a quartic curve which intersects the sphere $S$ in the four points of intersection of the four pairs of minimum lines belonging to the generation. It is known that there are ten systems of circles on a cyclide, and Laguerre has shown‡ that the other systems are obtained by transforming the tangent cones of the quadric drawn from points of the double curve of the developable circumscribing $S$ and the quadric.

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* *Œuvres*, p. 53.

† See Laguerre, *Œuvres*, p. 53.

‡ *Œuvres*, vol. II, p. 57.

Three non-intersecting lines determine a hyperboloid and therefore three non-intersecting circles orthogonal to the same sphere determine a cyclide uniquely. Any cyclide except Dupin's can appear as the surface common to three linear complexes if the quadric has a particular relation to $S$. The surfaces common to three linear complexes must have two distinct generations of circles, however; e. g., the binodal cyclide can be one if it is obtained by transforming a quadric which is tangent to $S$ at two points, but not if it is considered as the surface into which the tangent lines to a conic are transformed. The cyclide may degenerate into two pencils of circles which have a circle in common.

**5. The quadratic complex.** A quadratic complex of lines is transformed into a quadratic complex of circles. The transformation is as follows:

| *Line complex.* | *Circle complex.* |
|---|---|
| Lines of the complex which pass through a point generate a cone called the complex cone. | Circles of the complex which lie on a sphere envelope a sphero-quartic called the complex quartic. |
| Lines which lie in the same plane envelope a conic called the complex conic. | Circles of the complex which pass through a point pair generate a binodal cyclide called the complex cyclide. |
| The surface of singularities is the locus of those points whose complex cone breaks up into two planes, or the envelope of those planes whose complex conic breaks up into two pencils. | The surface of singularities is the envelope of those spheres whose complex quartic breaks up into two pencils, or the locus of those point pairs whose complex cyclide breaks up into two spheres. |

The surface of singularities of the quadratic line complex is the Kummer surface; the surface of singularities of the quadratic circle complex is obtained by transforming this surface. This transformed surface is of order 8; it contains the circle at infinity as a four-fold line, and has eight cuspidal points on the circle at infinity. The curve of intersection of $S$ with the Kummer surface is a focal line on the surface; the remainder of the focal curve is the double line of the developable circumscribed to this curve and the circle at infinity. There are 16 spheres which have singular contact with the surface

along a sphero-quartic; these 16 spheres are grouped in sixes so that the centers of the six spheres of a group lie on a conic; the six spheres of a group have two points in common; six of these planes of centers pass through the center of each sphere. The surface has 16 pairs of conical points derived from the 16 singular tangent planes of the Kummer surface.

The surfaces of singularities of the special forms of the quadratic complex are obtained by transforming the corresponding Kummer surface.

APPENDIX.

## A. Transformation of a minimum plane.

If $(x, y, z, t)$ are the homogeneous coordinates of a point, the equation of a minimum plane is

$$Ax + By + Cz + Dt = 0,$$

where the coefficients satisfy the relation

$$A^2 + B^2 + C^2 = 0.$$

The pencil of spheres defined by this plane and $S$, whose equation is

$$S = x^2 + y^2 + z^2 - R^2t^2 = 0,$$

is

$$(1) \qquad l(x^2 + y^2 + z^2 - R^2t^2) + 2kt(Ax + By + Cz + Dt) = 0,$$

or, rewriting,

$$(2) \qquad (lx + kAt)^2 + (ly + kBt)^2 + (lz + kCt)^2 = l^2R^2t^2 - 2klDt^2.$$

The point spheres are given by the values of $l$ and $k$ which satisfy

$$(3) \qquad l^2R^2t^2 - 2klDt^2 = 0.$$

The roots of this equation are $l = 0$ and $k/l = R^2/2D$. From equation (1) we see that the point sphere corresponding to $l = 0$ is

$$t(Ax + By + Cz + Dt) = 0,$$

which degenerates into $t = 0$ and $Ax + By + Cz + Dt = 0$. Thus it is seen that in a pencil of spheres defined by a minimum plane and $S$, one

of the point spheres degrades into the minimum plane itself together with the plane at infinity. The centers of the spheres of the pencil are the points $x:y:z:t=kA:kB:kC:l$ (see equation 2), and when $l=0$ the point becomes $x:y:z=A:B:C$, $t=0$; i. e., the center of the degraded point sphere is the point where the minimum plane touches the circle at infinity. In this sense it is seen that a minimum plane taken with the plane at infinity is a point sphere and that its center is on the circle at infinity. The center of the other point sphere of the pencil is

$$\left(\frac{AR^2}{2D}, \frac{BR^2}{2D}, \frac{CR^2}{2D}\right)$$

which is seen to be on the line passing through $O$ and having direction cosines proportional to $A$, $B$, $C$. This is the line joining $O$ to the point where the minimum plane touches the circle at infinity.

### B. Transformation of the plane at infinity

The equation of the plane at infinity is

$$Ax+By+Cz+Dt=0, \quad \text{where} \quad A=B=C=0.$$

From equation (3) above the point spheres of the pencil determined by this plane and $S$ are given by the solutions of

$$R^2t^2l^2-2Dtkl=0,$$

which are $k/l=R^2/2D$ and $l=0$.

The center of the point sphere corresponding to $R^2/2D$ is $(0, 0, 0)$ [see equation (2) of appendix A]. The point sphere corresponding to $l=0$ is the plane at infinity counted twice. Its center is $(kA/l, kB/l, kC/l)$, which is illusory since $A=B=C=l=0$. This means that the center of the point sphere which degrades into the plane at infinity counted twice is any point whatever in the plane at infinity.

### C. Transformation of a point in the plane at infinity.

The sphere $S'$ into which any point $(\xi:\eta:\zeta:1)$ is transformed is

$$S'\equiv(x-\xi t)^2+(y-\eta t)^2+(z-\zeta t)^2-r^2t^2=0,$$

where $r^2 = \xi^2 + \eta^2 + \zeta^2 - R^2$ (since $S$ and $S'$ are orthogonal). Expanding and making use of the above relation, we have

$$x^2 + y^2 + z^2 - 2t(\xi x + \eta y + \zeta z) + R^2t^2 = 0,$$

which we may write

$$\frac{1}{\sqrt{\xi^2 + \eta^2 + \zeta^2}}\left[x^2 + y^2 + z^2 - 2t\,(\xi x + \eta y + \zeta z) + R^2t^2\right] = 0.$$

Now as the point $(\xi : \eta : \zeta : 1)$ recedes to infinity along a line passing through $O$,

$$\frac{\xi}{\sqrt{\xi^2 + \eta^2 + \zeta^2}},\quad \frac{\eta}{\sqrt{\xi^2 + \eta^2 + \zeta^2}},\quad \frac{\zeta}{\sqrt{\xi^2 + \eta^2 + \zeta^2}}$$

remain finite and the last equation reduces to

$$\frac{\xi x + \eta y + \zeta z}{\sqrt{\xi^2 + \eta^2 + \zeta^2}} = 0, \quad t = 0.$$

The sphere therefore reduces to a plane passing through $O$ perpendicular to the line whose direction cosines are proportional to $\xi, \eta, \zeta$, together with the plane at infinity.

### D. Transformation of a line in the plane at infinity.

For convenience take a line perpendicular to the $xy$-plane, whose equations are $x = \xi$, $y = \eta$. The circle into which it is transformed is

$$(x - \xi t)^2 + (y - \eta t)^2 - r^2t^2 = 0,\ z = 0,$$

where $R^2 = \xi^2 + \eta^2 - r^2$, since the circle is orthogonal to $S$. Expanding and making use of this relation, we have

$$(4) \qquad x^2 + y^2 - 2t\,(\xi x + \eta y) + R^2t^2 = 0, \quad z = 0,$$

$$(5) \qquad \frac{1}{\sqrt{\xi^2 + \eta^2}}\left[x^2 + y^2 - 2t(\xi x + \eta y) + R^2t^2\right] = 0, \quad z = 0.$$

Now as the line recedes to infinity in a definite direction, $\frac{\xi}{\sqrt{\xi^2 + \eta^2}}$ and $\frac{\eta}{\sqrt{\xi^2 + \eta^2}}$ remain constant and the last equation reduces to

$$\frac{\xi x + \eta y}{\sqrt{\xi^2 + \eta^2}} = 0, \quad t = 0; \quad z = 0,$$

which shows that the circle reduces to the lines where the plane $z = 0$ cuts the planes $\xi x + \eta y = 0$, and $t = 0$; i. e., it is transformed into a line through $O$ together with a line in the plane at infinity.

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